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Retardation effects in the rotating string model

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Abstract

A new method to study the retardation effects in mesons is presented. Inspired from the covariant oscillator quark model, it is applied to the rotating string model in which a non zero value is allowed for the relative time between the quark and the antiquark. This approach leads to a retardation term which behaves as a perturbation of the meson mass operator. It is shown that this term preserves the Regge trajectories for light mesons, and that a satisfactory agreement with the experimental data can be obtained if the quark self-energy contribution is added. The consequences of the retardation on the Coulomb interaction and the wave function are also analyzed.

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I. INTRODUCTION

The retardation effect between two interacting particles is a relativistic phenomenon, due to the finiteness of the interaction speed. The light mesons are typical systems in which the retardation mechanisms can significantly contribute to the dynamics, since the light quark can move at a speed close to the speed of light. To take into account this effect into effective models, one has to work with a fully covariant theory. The most elegant approach, from a theoretical point of view, is the formalism with constraints [1, 2], but it quickly leads to complex equations, uneasy to deal with if one wants to get analytical or numerical results. Another covariant approach of mesons is the covariant oscillator quark model (COQM), which allows to find an analytical expression for the wave function and numerical results in good agreement with the data [3, 4]. Unfortunately, this model uses a quadratic potential, this form being different from the linear potential, commonly assumed to describe the confining part of the interaction and suggested by lattice calculations. Moreover, vanishing quark masses are not allowed in this approach.

Apart from these two approaches, most of the effective models found in the literature are based on the equal time ansatz, which simply takes the time coordinates of both particles to be equal, neglecting the retardation effects. This procedure allows to deal with simpler equations, and relativistic corrections can be obtained by developing an expansion in v^2/c^2 of the model considered, like the Bethe-Salpeter equation or even the QCD Lagrangian [5, 6].

The model we propose here is an attempt to include retardation effects into the rotating string model (RSM) [7, 8] without making such an expansion, in order to estimate the retardation contribution for light quarks. The RSM is an effective model derived from the QCD Lagrangian, describing a meson by two spinless quarks linked by a straight string. It has been shown that the RSM was classically equivalent to the relativistic flux tube model [9, 10]. This last model, firstly presented in Ref. [11, 12], yields meson spectra in good agreement with the experimental data [13]. Our method, inspired from the COQM, relies on the hypothesis that the relative time between the quark and the antiquark must have a non zero value. In our framework, the evolution parameter of the system is not the common proper time of the quarks and the string, but the time coordinate of the center of mass which plays the role of an "average" time.

Our paper is organized as follows. In Sec. II, we present the general formalism of our

approach. We compute the retardation contribution to the meson mass in Sec. III. We find different approximations for this contribution in Sec. IV, with a special interest for light quarks, and we numerically study the retardation effects in Sec. V. As our model relies on unusual hypothesis, it is worth comparing our results with those of other existing models. This is done in Sec. VI. Finally, we compare the meson spectra of our RSM including the retardation term with the experimental data in Sec. VII. Some concluding remarks are given in Sec. VIII. The appendix contains some useful formulas.

II. THE ROTATING STRING MODEL WITH NON ZERO RELATIVE TIME

It has been shown that, starting from the QCD Lagrangian and neglecting the spin contribution of the quark and the antiquark, the Lagrange function of a meson can be built from a Nambu-Goto action [7] which reads ($\eta = \text{diag}(+---)$ and $\hbar = c = 1$)

$$\mathcal{L}(\tau) = -m_1\sqrt{\dot{\boldsymbol{x}}_1^2} - m_2\sqrt{\dot{\boldsymbol{x}}_2^2} - a\int_0^1 d\beta\sqrt{(\dot{\boldsymbol{w}}\boldsymbol{w}')^2 - \dot{\boldsymbol{w}}^2\boldsymbol{w}'^2}. \tag{1}$$

The two first terms are the kinetic energy operators of the quark and the antiquark, whose current masses are m_1 and m_2. These two particles are attached by a string with a tension a. $\boldsymbol{x}_i$ is the coordinate of the quark i and $\boldsymbol{w}$ is the coordinate of the string. $\boldsymbol{w}$ depends on two variables defined on the string worldsheet: One is spacelike, β, and the other timelike, τ. Derivatives are denoted $\boldsymbol{w}' = \partial_\beta \boldsymbol{w}$ and $\dot{\boldsymbol{w}} = \partial_\tau \boldsymbol{w}$. In this picture, τ is a common proper time for the string and the quarks. Introducing auxiliary fields to get rid of the square root in the Lagrangian (1) and making the straight line ansatz to describe the string, an effective Lagrangian can be derived [8]

$$\begin{aligned}\mathcal{L} \;=\; -\frac{1}{2}\Bigg[&\frac{m_1^2}{\mu_1} + \frac{m_2^2}{\mu_2} + a_1\dot{\boldsymbol{R}}^2 + 2a_2\dot{\boldsymbol{R}}\dot{\boldsymbol{r}} - 2(c_1 + \dot{\zeta}a_1)\dot{\boldsymbol{R}}\boldsymbol{r}\\ &-2(c_2 + \dot{\zeta}a_2)\dot{\boldsymbol{r}}\boldsymbol{r} + a_3\dot{\boldsymbol{r}}^2 + (a_4 + 2\dot{\zeta}c_1 + \dot{\zeta}^2 a_1)\boldsymbol{r}^2\Bigg],\end{aligned} \tag{2}$$

where the coefficients a_1, a_2, ..., given in the appendix, depend on various auxiliary fields μ_1, μ_2, ν, and η. The parameter ζ defines the position $\boldsymbol{R}$ of the center of mass: $\boldsymbol{R} = \zeta\boldsymbol{x}_1 + (1-\zeta)\boldsymbol{x}_2$. $\boldsymbol{r}$ is the relative coordinate $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$. The auxiliary fields μ_1 and μ_2 are seen as constituent masses for the quarks, and ν can be interpreted in the same way as an effective energy for the string whose "static" energy is ar [9, 10]. Let us note that the straight line ansatz for the string implies $\boldsymbol{w} = \boldsymbol{R} + (\beta - \zeta)\boldsymbol{r}$.

The total and relative momentum, computed from the Lagrangian (2), are respectively

$$P_\mu = \frac{\partial \mathcal{L}}{\partial \dot{R}^\mu} = -a_1 \dot{R}_\mu - a_2 \dot{r}_\mu + (c_1 + \dot{\zeta} a_1) r_\mu, \tag{3a}$$

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{r}^\mu} = -a_2 \dot{R}_\mu - a_3 \dot{r}_\mu + (c_2 + \dot{\zeta} a_2) r_\mu. \tag{3b}$$

As we will work in the center of mass frame, the total vector momentum $\vec{P}$ of the system must vanish, which implies that

$$\dot{\vec{R}} = \frac{(c_1 + \dot{\zeta} a_1)\vec{r} - a_2 \dot{\vec{r}}}{a_1}. \tag{4}$$

Moreover, the relative vector momentum $\vec{p}$ is given by

$$\vec{p} = a_2 \dot{\vec{R}} + a_3 \dot{\vec{r}} - (c_2 + \dot{\zeta} a_2)\vec{r}. \tag{5}$$

Thus, we impose $a_2 = 0$ in order that $\vec{p}$ does not depend on the motion of the center of mass. This leads to the following value for the parameter ζ

$$\zeta = \frac{\mu_1 + \int_0^1 d\beta\, \beta\, \nu}{\mu_1 + \mu_2 + \int_0^1 d\beta\, \nu}. \tag{6}$$

Equation (6) reduces to $\zeta = 1/2$ in the symmetrical case ($m_1 = m_2$) and to $\zeta = m_1/(m_1+m_2)$ in the nonrelativistic limit, as expected [10].

To go a step further, one usually takes the temporal coordinates of the quarks and the string to be equal to the common proper time τ, this time being also the time t in the center of mass frame

$$x_1^0 = x_2^0 = w^0 = \tau = t. \tag{7}$$

Then, we have $\boldsymbol{r} = (0, \vec{r}\,)$, $\boldsymbol{R} = (t, \vec{R}\,)$, $\dot{\boldsymbol{r}} = (0, \dot{\vec{r}}\,)$, and $\dot{\boldsymbol{R}} = (1, \dot{\vec{R}}\,)$. This procedure allows to deal with simpler equations, but neglects the relativistic retardation effects due to a possible non zero value of the relative time r^0. Since these effects are precisely those we want to study in this paper, we have to make a less restrictive hypothesis. As in the formalism of the COQM [3, 4], we define

$$\boldsymbol{r} = (\sigma, \vec{r}\,), \;\; \boldsymbol{R} = (\bar{t}, \vec{R}\,). \tag{8}$$

The temporal coordinate of the center of mass, $\bar{t}$, can be seen as an "average time" for the meson. This is particularly clear in the symmetrical case, where $\bar{t} = (x_1^0 + x_2^0)/2$. Our choice is to take $\bar{t}$ as the evolution parameter of the system. We identify it as the common proper

time for the quarks and the string, and the dotted quantities are derived with respect to $\bar{t}$. We have for example

$$\dot{\boldsymbol{r}} = (\dot{\sigma}, \dot{\vec{r}}), \;\; \dot{\boldsymbol{R}} = (1, \dot{\vec{R}}). \tag{9}$$

The special case $\sigma = 0$ is equivalent to the relation (7).

The Lagrangian (2) can now be rewritten using formulas (4), (5), (8), and (9) as

$$\mathcal{L} = \mathcal{L}_0 + \Delta\mathcal{L}, \tag{10}$$

with

$$\mathcal{L}_0 = -\frac{1}{2}\left[\frac{m_1^2}{\mu_1} + \frac{m_2^2}{\mu_2} + a_1 + \frac{1}{a_1}\left((c_1^2 - a_4 a_1)\vec{r}^{\,2} - a_3 a_1 \dot{\vec{r}}^{\,2} + 2a_1 c_2 \dot{\vec{r}}\,\vec{r}\right)\right], \tag{11}$$

$$\Delta\mathcal{L} = (c_1 + \dot{\zeta} a_1)\sigma + c_2 \dot{\sigma}\sigma - \frac{a_3}{2}\dot{\sigma}^2 - \frac{1}{2}(a_4 + 2\dot{\zeta} c_1 + \dot{\zeta}^2 a_1)\sigma^2. \tag{12}$$

We have gathered the relative time dependent terms in $\Delta\mathcal{L}$, which contains the contribution of the retardation. Let us remark that the Lagrangian $\mathcal{L}_0$ does not depend on $\dot{\zeta}$.

We shall consider in the following $\Delta\mathcal{L}$ as a perturbation of $\mathcal{L}_0$. With this hypothesis, the auxiliary fields can be eliminated by considering only the constraint $\delta\mathcal{L}_0 = 0$. In Ref. [10], it is shown how a set of three equations can be derived from the Lagrangian (11), to define the usual rotating string model (RSM)

$$0 = \mu_1 y_1 - \mu_2 y_2 - \frac{ar}{y_t}\left(\sqrt{1-y_1^2} - \sqrt{1-y_2^2}\right), \tag{13a}$$

$$\frac{L}{r} = \frac{1}{y_t}(\mu_1 y_1^2 + \mu_2 y_2^2) + \frac{ar}{y_t^2}(F(y_1) + F(y_2)), \tag{13b}$$

$$H_0 = \frac{1}{2}\left[\frac{p_r^2 + m_1^2}{\mu_1} + \frac{p_r^2 + m_2^2}{\mu_2} + \mu_1(1+y_1^2) + \mu_2(1+y_2^2)\right] + \frac{ar}{y_t}(\arcsin y_1 + \arcsin y_2), \tag{13c}$$

with

$$F(y_i) = \frac{1}{2}\left[\arcsin y_i - y_i\sqrt{1-y_i^2}\right] \quad \text{and} \quad y_t = y_1 + y_2. \tag{13d}$$

p_r is the radial momentum and y_i can be seen as the transverse velocities of the quark i. The first relation gives the cancellation of the total momentum in the center of mass frame, while the two last ones define respectively the angular momentum and the Hamiltonian. As we can see in Eqs. (13), the only remaining auxiliary fields are μ_i. The extremal values of

the auxiliary fields η and ν are given by [10]

$$\eta_0 = \kappa\,(\beta - \phi)\,, \tag{14a}$$

$$\nu_0 = \frac{ar}{\sqrt{1 - y_t^2(\beta - \zeta)^2}}, \tag{14b}$$

with $\kappa = -\frac{\vec{p}\cdot\vec{r}}{\tilde{\mu}r^2}$, $\phi = \frac{\mu_1}{\mu_1+\mu_2}$, and $\tilde{\mu} = \frac{\mu_1\mu_2}{\mu_1+\mu_2}$. Let us remark that a closed form can not be obtained for the Hamiltonian H_0 because it impossible to eliminate analytically the variables y_1 and y_2 as a function of L, by means of the two first Eqs. (13).

Since we have a contribution from the relative time, the total Hamiltonian is given by

$$H = H_0 + \Delta H, \tag{15}$$

with

$$\Delta H = \Sigma\dot{\sigma} - \Delta\mathcal{L}. \tag{16}$$

With $\Sigma = \partial\mathcal{L}/\partial\sigma$, Eq. (3b) leads to

$$\dot{\sigma} = \frac{c_2\sigma - \Sigma}{a_3}, \tag{17}$$

and finally we obtain

$$\Delta H = -\frac{\Sigma^2}{2a_3} + \frac{c_2}{a_3}\Sigma\sigma - (c_1 + \dot{\zeta}a_1)\sigma - \frac{c_2^2}{2a_3}\sigma^2 + \frac{1}{2}(a_4 + 2\dot{\zeta}c_1 + \dot{\zeta}^2 a_1)\sigma^2, \tag{18}$$

the perturbation of the RSM Hamiltonian due to the retardation effect.

In the following, for simplicity, we will focus on the symmetrical case. Then $\dot{\zeta} = 0$, and the Hamiltonian (18) becomes

$$\Delta H = -\frac{1}{2a_3}\left[\Sigma^2 - 2c_2\Sigma\sigma + (c_2^2 - a_4a_3)\sigma^2\right]. \tag{19}$$

Wether ΔH is really a perturbation or not has to be verified. We will check this hypothesis a posteriori by a numerical computation of the retardation contribution to the meson masses (see Sec. V).

III. RETARDATION EFFECTS

A. Contribution to the mass

Up to now, we were working in a classical framework. But in order to study the influence of the retardation on the meson spectrum, we have to consider a quantized version of the

total RSM Hamiltonian $H_0 + \Delta H$. We can thus replace L by $\sqrt{\ell(\ell+1)}$ and consider r and σ as operators such that

$$[r,\, p_r] = i,\ \ [\sigma,\, \Sigma] = -i. \tag{20}$$

The total Hamiltonian has schematically the following structure

$$H(\sigma, r) = H_0(r) + \Delta H(\sigma, r). \tag{21}$$

The relative time σ only appears in the perturbation, and H_0 only depends on the radius r. So, we make the following ansatz to write the total wave function

$$|\psi(\boldsymbol{r})\rangle = |R(\vec{r})\rangle \otimes |A(\sigma)\rangle\,, \tag{22}$$

where $|R(\vec{r})\rangle$ is a solution of the eigenequation

$$H_0(r)\,|R(\vec{r})\rangle = M_0\,|R(\vec{r})\rangle\,. \tag{23}$$

Such a problem can be solved, for instance, by the Lagrange mesh technique [14]. The total mass is written

$$\begin{aligned} M &= M_0 + \langle A(\sigma)| \otimes \langle R(\vec{r})|\, \Delta H(r,\sigma)\, |R(\vec{r})\rangle \otimes |A(\sigma)\rangle \\ &= M_0 + \Delta M. \end{aligned} \tag{24}$$

the contribution ΔM is then given by the solution of the eigenequation

$$\Delta\mathcal{H}(\sigma)\,|A(\sigma)\rangle = \Delta M\,|A(\sigma)\rangle\,, \tag{25}$$

where

$$\Delta\mathcal{H}(\sigma) = \langle R(\vec{r})|\, \Delta H(r,\sigma)\, |R(\vec{r})\rangle\,. \tag{26}$$

In order to eliminate the unphysical excitations of the relative time, we consider only the ground state of the Hamiltonian $\Delta\mathcal{H}(\sigma)$, as it is done in Refs. [3, 4]. Using formula (19), this Hamiltonian is defined by

$$\Delta\mathcal{H} \approx -\frac{1}{2\langle a_3\rangle}\left[\Sigma^2 - \langle c_2\rangle\{\Sigma,\sigma\} + \langle c_2^2 - a_4 a_3\rangle\sigma^2\right], \tag{27}$$

where all mean values $\langle\rangle$ are computed with a space function $R(\vec{r})$ and where $\{A, B\} = AB + BA$. We also use the approximation $\langle 1/x\rangle \approx 1/\langle x\rangle$.

Using Eqs. (14a) and (A.3), we see that $c_2 \propto \kappa$. Since we are in the symmetrical case, we can assume $\langle \kappa \rangle = 0$, and so $\langle c_2 \rangle = 0$. On the contrary, $\langle c_2^2 \rangle \neq 0$ because $\langle p_r^2 \rangle > 0$. Finally the Hamiltonian $\Delta \mathcal{H}$ takes its final form

$$\Delta \mathcal{H} = -\frac{1}{2\langle a_3 \rangle} \left[\Sigma^2 + \langle c_2^2 - a_4 a_3 \rangle \sigma^2 \right]. \tag{28}$$

With our approximations, the retardation contribution to the Hamiltonian looks like an harmonic oscillator for the canonical variables (σ, Σ). Let us define

$$\rho^2 = c_2^2 - a_4 a_3. \tag{29}$$

Thanks to formulas (A.1), we can compute ρ^2

$$\begin{aligned} \rho^2 &= \frac{a}{2ry} \left(\frac{\mu}{2} + \frac{ar}{8y^3} \left(-y\sqrt{1-y^2} + \arcsin y \right) \right) \left(y\sqrt{1-y^2} + \arcsin y \right) \\ &- \frac{ap_r^2}{4\mu r y^3} \left(-y\sqrt{1-y^2} + \arcsin y \right). \end{aligned} \tag{30}$$

Assuming that $\langle \rho^2 \rangle > 0$ (this is checked in Sec. IV), the ground state solution of the eigenequation (25) is given by

$$\Delta M = -\frac{1}{2}\omega, \tag{31a}$$

$$A(\sigma) = \left(\frac{\beta}{\pi} \right)^{1/4} \exp\left(-\frac{\beta}{2}\sigma^2 \right), \tag{31b}$$

with

$$\beta = \sqrt{\langle \rho^2 \rangle}, \tag{31c}$$

$$\omega = \frac{\beta}{\langle a_3 \rangle}. \tag{31d}$$

An immediate conclusion to draw from Eqs. (31) is that the retardation effects bring a negative contribution to the meson masses, and that the more probable value for the relative time is $\sigma = 0$. It is worth noting that these results are formally identical to those of the COQM. In Sec. VI, we compare the two approaches with more details.

B. Modification of the coulomb term

In the RSM, the string contribution takes into account the interactions at large distances, which are responsible for the confinement. To make more realistic models, it is necessary

to add short range potentials [13]. For instance, the one gluon exchange mechanism can be simulated by a Coulomb term

$$V_C(r) = -\frac{4}{3}\frac{\alpha_S}{r}, \tag{32}$$

with α_S the strong coupling constant. This formula must be modified if we consider the retardation effects. Indeed, the quark and the antiquark are able to exchange one gluon if their separation $\boldsymbol{r}$ is light-like, that is to say if $\sigma^2 - r^2 = 0$. The probability for σ to be negative or positive is

$$p\ (\sigma < 0) = \int_{-\infty}^{0} d\sigma A(\sigma)^2 = \frac{1}{2} = p\,(\sigma > 0). \tag{33}$$

Consequently, as in the COQM [3, 4], we make the following substitution

$$V_C \to V_C \frac{1}{2\lambda}\left[\delta(\sigma + r) + \delta(\sigma - r)\right], \tag{34}$$

where λ is an energy scale which is introduced so as to give the correct dimension. This parameter is purely phenomenological and could depend on the quark masses [3]. In this paper, we assume that λ is a constant. The effective Coulomb potential $\tilde{V}_C$, treated as a perturbation, is then computed with the relation

$$\tilde{V}_C = \int_{-\infty}^{+\infty} d\sigma A(\sigma)^2 V_C \frac{1}{2\lambda}\left[\delta(\sigma + r) + \delta(\sigma - r)\right], \tag{35}$$

and we obtain a damped effective Coulomb potential,

$$\tilde{V}_C = -\frac{4}{3}\frac{\alpha_S}{\lambda r}\left(\frac{\beta}{\pi}\right)^{1/2} \exp\left(-\beta r^2\right). \tag{36}$$

IV. APPROXIMATIONS FOR ΔM

The key ingredient to compute the retardation term (31a) is the knowledge of ρ^2 and a_3, respectively given by formulas (30) and (A.3). As these expressions are complicated, we will try to get various simpler ones following the value of the quark mass m. In the following we will use two limiting cases for a_3 and ρ^2 quantities: $y = 0$ corresponds to a vanishing angular momentum or to a very high mass, and $y = 1$ correspond to a very high angular momentum or to a very small mass. Both situations are summarized in Table I.

The solutions of the RSM equations are, in good approximation, very close to the solutions of a two-body spinless Salpeter equation with a linear potential

$$H^{SS} = 2\sqrt{\vec{p}^{\,2} + m^2} + ar \tag{37}$$

TABLE I: Values of a_3 and ρ^2 for $y=0$ and $y=1$.

	$y=0$	$y=1$
a_3	$\dfrac{\mu}{2}+\dfrac{ar}{12}$	$\dfrac{\mu}{2}+\dfrac{\pi ar}{16}$
ρ^2	$\dfrac{a\mu}{2r}+\dfrac{a^2}{12}-\dfrac{ap_r^2}{6\mu r}$	$\dfrac{\pi a\mu}{8r}+\dfrac{\pi^2 a^2}{64}-\dfrac{\pi ap_r^2}{8\mu r}$

and with the pure string correction treated as a perturbation [15]. For massless quarks, this correction is only about 6% of the meson mass [16]. So, we will work with the Hamiltonian H^{SS} without the string correction, except in Sec. VII. Actually, this approximation is equivalent to consider the RSM at the order y^2 [16]. Within this framework, the extremal value of the auxiliary field μ is [9, 16]

$$\mu = \sqrt{\vec{p}^{\,2}+m^2}. \tag{38}$$

We have then

$$\langle p_r^2\rangle \approx \langle\mu\rangle^2 - m^2 - \frac{\ell(\ell+1)}{\langle r\rangle^2}, \tag{39}$$

and $0 < \langle p_r^2\rangle \le \langle\mu\rangle^2$.

A. High quark mass

If we assume that $\langle\mu\rangle \gg a\langle r\rangle$, we can set $y\approx 0$. In this case, the relations (31a) and (31c) with $\langle\mu\rangle \gg a\langle r\rangle$ reduces to

$$\beta_h \approx \sqrt{\frac{a\langle\mu\rangle}{2\langle r\rangle}}\sqrt{1-\frac{\langle p_r^2\rangle}{3\langle\mu\rangle^2}}, \tag{40}$$

$$\Delta M_h \approx -\sqrt{\frac{a}{2\langle\mu\rangle\langle r\rangle}}\sqrt{1-\frac{\langle p_r^2\rangle}{3\langle\mu\rangle^2}}. \tag{41}$$

It is clear that $\langle\rho^2\rangle$ is positive. If m is very large, the dynamical effects can be neglected and, using $\langle\mu\rangle\approx m$, we have

$$\beta_{hh} \approx \sqrt{\frac{am}{2\langle r\rangle}}, \tag{42}$$

$$\Delta M_{hh} \approx -\sqrt{\frac{a}{2m\langle r\rangle}}. \tag{43}$$

The eigenvalues M_0 are then given with a good accuracy by [17]

$$M_0 \approx 2m + \left(\frac{a^2}{m}\right)^{1/3} \epsilon_{n\ell}, \tag{44}$$

where $\epsilon_{n\ell}$ is an eigenvalue of the dimensionless Hamiltonian $(\vec{q}^{\,2} + |\vec{x}\,|)$. The nonrelativistic virial theorem implies that

$$a\langle r\rangle = \frac{2}{3}(M_0 - 2m) = \frac{2}{3}\left(\frac{a^2}{m}\right)^{1/3} \epsilon_{n\ell}. \tag{45}$$

So we obtain

$$\beta_{hh} \approx \frac{(am)^{2/3}}{2}\sqrt{\frac{3}{\epsilon_{n\ell}}}, \tag{46}$$

$$\Delta M_{hh} \approx -\frac{1}{2}\left(\frac{a^2}{m}\right)^{1/3}\sqrt{\frac{3}{\epsilon_{n\ell}}}. \tag{47}$$

Approximate values for the quantities $\epsilon_{n\ell}$ can be found in Ref. [16].

B. Vanishing quark mass

In this section, we will work with the Hamiltonian H^{SS} for $m = 0$. With these conditions, the relativistic virial theorem [18] gives the following results [9]

$$M_0 \approx \langle H^{SS}\rangle = 4\langle\mu\rangle, \tag{48}$$

$$a\langle r\rangle = 2\langle\mu\rangle. \tag{49}$$

Let us first focus on the case $\ell = 0$, for which $y = 0$ and $\langle p_r^2\rangle \approx \langle\mu\rangle^2$. Thanks to these relations, we have for light quarks

$$\beta_l|_{\ell=0} = \frac{a}{2}, \tag{50}$$

$$\Delta M_l|_{\ell=0} = -\frac{3a}{2M_0}. \tag{51}$$

Secondly, let us consider the case $\ell \gg 1$, for which we can assume that $y = 1$. We then find

$$\beta_l|_{\ell\gg1} = \frac{a}{4}\sqrt{\pi\left(1 + \frac{\pi}{4} - \frac{16\langle p_r^2\rangle}{M_0^2}\right)}, \tag{52}$$

$$\Delta M_l|_{\ell\gg1} = -\frac{a}{M_0}\left(\frac{4}{4+\pi}\right)\sqrt{\pi\left(1 + \frac{\pi}{4} - \frac{16\langle p_r^2\rangle}{M_0^2}\right)}. \tag{53}$$

Combining Eqs. (39), (48), and (49), we find

$$\langle p_r^2 \rangle \approx \langle \mu \rangle^2 - \frac{64a^2\ell(\ell+1)}{M_0^4}. \tag{54}$$

Consequently, for massless quarks with high angular momentum, we have

$$\beta_l|_{\ell\gg 1} = \frac{a}{4}\sqrt{\pi\left(\frac{\pi}{4} + \frac{64a^2\ell(\ell+1)}{M_0^4}\right)}, \tag{55}$$

$$\Delta M_l|_{\ell\gg 1} = -\frac{a}{M_0}\left(\frac{4}{4+\pi}\right)\sqrt{\pi\left(\frac{\pi}{4} + \frac{64a^2\ell(\ell+1)}{M_0^4}\right)}. \tag{56}$$

Again, $\langle \rho^2 \rangle$ is positive for $\ell = 0$ and $\ell \gg 1$.

In good approximation, it appears that

$$\Delta M_l \propto \frac{1}{M_0}. \tag{57}$$

The square meson mass composed of light quarks is then given by

$$M_l^2 \approx M_0^2 + 2M_0\, \Delta M_l, \tag{58}$$

where the term ΔM_l^2 is neglected. This shows that the retardation term only causes a global shift of the square meson masses and consequently preserves the Regge trajectories, since $M_0^2 \propto \ell$ for large values of ℓ [15, 16].

V. NUMERICAL RESULTS

In order to obtain better values for the contribution of the retardation, we will compute it at the second order in y. In this case, the quantities a_3 and ρ^2 become

$$\langle a_3 \rangle \approx \left(\frac{a\langle r \rangle}{12} + \frac{\langle \mu \rangle}{2}\right) + \frac{1}{40}a\langle r \rangle\langle y^2 \rangle, \tag{59a}$$

$$\langle \rho^2 \rangle \approx \left(\frac{a^2}{12} - \frac{a\langle p_r^2 \rangle}{6\langle \mu \rangle\langle r \rangle} + \frac{a\langle \mu \rangle}{2\langle r \rangle}\right) + \left(\frac{a^2}{90} - \frac{a\langle p_r^2 \rangle}{20\langle \mu \rangle\langle r \rangle} - \frac{a\langle \mu \rangle}{12\langle r \rangle}\right)\langle y^2 \rangle. \tag{59b}$$

These expressions can be calculated using the relation [16]

$$\langle y^2 \rangle \approx \frac{\ell(\ell+1)}{\langle r \rangle^2\left(a\langle r \rangle/6 + \langle \mu \rangle\right)^2}. \tag{59c}$$

In the following, the retardation contribution obtained thanks to Eqs. (59) will be called "exact" by opposition to the more approximate formulas obtained in Sec. IV, and it will

be simply denoted by ΔM. The exact contribution is compared to the approximate ones in Fig. 1 ($\langle \rho^2 \rangle$ is always positive). In this graph, $\ell = n = 0$, but the qualitative features of the curves are the same for other quantum numbers. We can thus determine a validity domain for each approximation. In fact, ΔM_l is the best for $m < 0.175$ GeV (u, d quarks, which are commonly denoted n). For 0.175 GeV $< m <$ 4.0 GeV (s, c quarks), ΔM_h is rather good, and for masses larger than 4.0 GeV (b quark), ΔM_{hh} works well. As expected, the retardation contribution is less important when the quark mass is larger, for which a nonrelativistic treatment is more justified. One could ask how the systematic substitution $\langle \mu^2 \rangle \to \langle \mu \rangle^2$ does affect the results. Actually, the values obtained by the two methods differ at most by 5%. So, we will maintain our choice, which is to use systematically powers of $\langle \mu \rangle$.

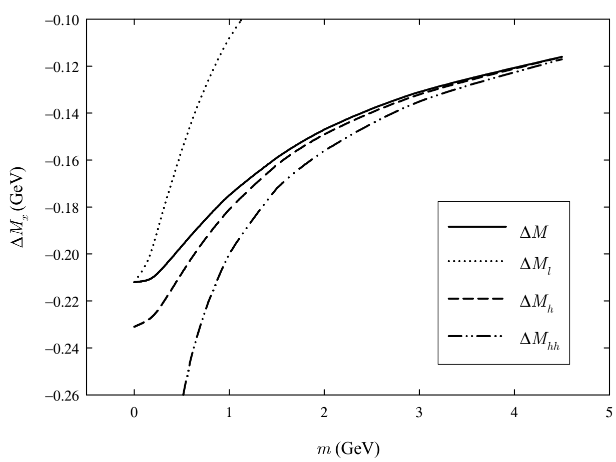

FIG. 1: Comparison between the exact retardation term and the different approximations of Sec. IV for various quark masses, with $a = 0.2$ GeV^2 and $\ell = n = 0$.

We focus now on the light quarks, especially the massless case: The retardation effects are indeed expected to be the largest when $m = 0$, for which the motion is the most relativistic. Typical behaviors of ΔM with ℓ and n are showed in Fig. 2. If we take $a = 0.2$ GeV^2, the ground state mass M_0 is 1.413 GeV. The contribution of the retardation is −0.205 GeV. So, in the worst case, the contribution is about 15% of the non perturbed mass. This result justifies a posteriori the perturbative theory we built in Secs. II and III. Moreover, we see that, for a fixed quark mass, the retardation contribution decreases when the different quantum numbers, ℓ or n, increase. This means that the key element is not the quark mass m, but the its constituent mass $\langle \mu \rangle$ [19] which also increases with these quantum numbers.

The more the constituent mass is large, the more the retardation effect is small.

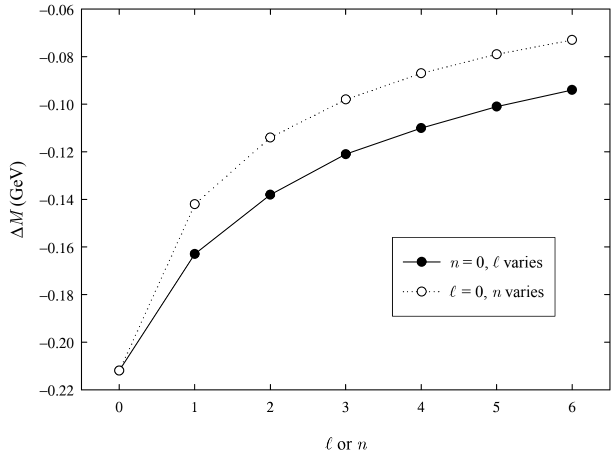

FIG. 2: Exact retardation term versus: ℓ for $n = 0$ (filled circles) and n for $\ell = 0$ (empty circles), with $a = 0.2$ GeV^2 and $m = 0$.

In Sec. IV, we showed that the retardation only causes a global shift of the Regge trajectories for light quarks. Even if this result is only approximate, we can check in Fig. 1 that we can have confidence in our formula ΔM_l when $m = 0$. As a supplementary test in this case, we have plotted in Fig. 3 the square meson masses versus the angular momentum, with and without the exact retardation term. We clearly see that the linearity of the Regge trajectories is preserved as well as the slope.

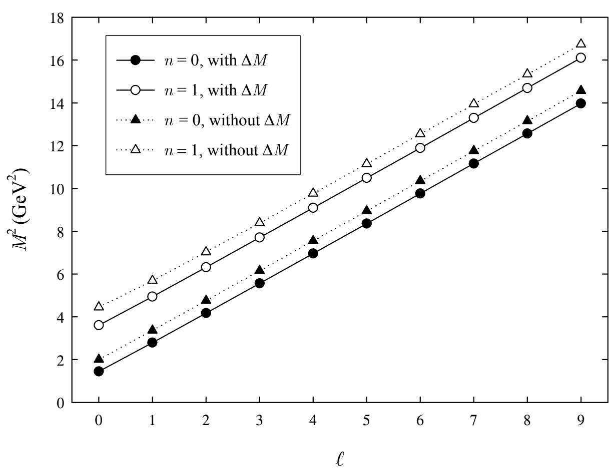

FIG. 3: Regge trajectories with exact retardation term (circles) and without (triangles), for $n = 0$ and 1, with $m = 0$ and $a = 0.2$ GeV^2.

Another interesting quantity to study is β, given by Eq. (31c), the size of the relative

time part of the wave function (31b) and the range of the effective Coulomb potential (36). Let us see what happens for two extreme cases: the n quark and the b quark, for which we take $m_n = 0$ and $m_b = 4.660$ GeV. For $n = \ell = 0$, β_l, used for the n quark, is given by formula (50). For the b quark, we can use formula (46). For a standard value $a = 0.2$ GeV^2, we find

$$\beta_{hh}\big|_{\ell=n=0} = 0.542 \text{ GeV}^2 \gg \beta_l\big|_{\ell=n=0} = 0.1 \text{ GeV}^2. \tag{60}$$

So, the wave function is much more peaked around $\sigma = 0$ when the mass increases. The numerical evaluation with Eqs. (59) gives

$$\beta\big|_{\ell=n=0} = 0.533 \text{ GeV}^2 \gg \beta\big|_{\ell=n=0} = 0.096 \text{ GeV}^2, \tag{61}$$

result very close to the approximate one (60).

The parameter β also considerably affects the Coulomb potential through Eq. (36), as it is shown in Fig. 4. We will see in Sec. VII that the effective potential can become very small with respect to the retardation term.

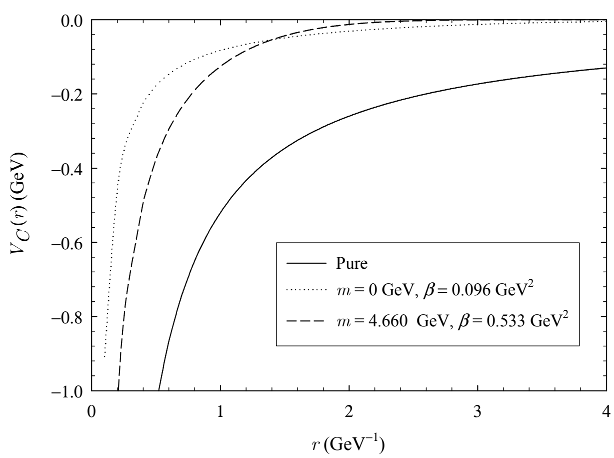

FIG. 4: Pure Coulomb potential with $\alpha_S = 0.4$ (straight line), and effective Coulomb potentials for $m = 0$ and $m = 4.660$ GeV with $\lambda = 1$ GeV (dotted lines).

VI. COMPARISON WITH OTHER APPROACHES

Since the relative time is introduced in the same way for our model and the COQM, both approaches share some common properties. For instance, the relative time part of our

wave function Eq. (31b) and the counterpart in the COQM have the same form. In this last model, for two quarks with the same mass m, we have [3, 4]

$$A(\sigma) = \left(\frac{\kappa}{\pi}\right)^{1/4} \exp\left(-\frac{\kappa}{2}\sigma^2\right) \quad \text{with} \quad \kappa = \sqrt{\frac{mK}{2}}, \tag{62}$$

where K is a constant related to the interquark potential

$$U = -\frac{1}{2}K\boldsymbol{r}^2. \tag{63}$$

Moreover this model predicts also linear Regge trajectories, with a constant square mass shift ΔM^2_{COQM} due to the retardation which is given by

$$\Delta M^2_{\text{COQM}} = -2\sqrt{2mK}. \tag{64}$$

Besides this formal similarities, the physical content of these two models are nevertheless very different: A comparison is made in Table II.

TABLE II: Comparison between the COQM and our model.

	COQM	**Our model**
Mass operator	square mass operator	ordinary Hamiltonian
Confinement	quadratic	linear
Allowed masses	non zero	all
Treatment of retardation	exact	in perturbation
Time and space decoupling	complete and exact	partial and approximate
Shift in energy	negative and constant	negative and state dependent
Relative time wave function	gaussian with constant size	gaussian with state dependent size

A more usual method to get relativistic corrections (including retardation) is to consider the v^2/c^2 terms of the Bethe-Salpeter equation or of the Wilson loop formulation of QCD. This was done for example in Refs. [5, 6]. In Ref. [6], the relativistic correction to the linear confinement potential due to the retardation is given by

$$\Delta M = -\left\langle \frac{a}{m^2}\left(\frac{\ell(\ell+1)}{2r} + rp_r^2\right)\right\rangle. \tag{65}$$

As this correction term is obtained by a expansion in v^2/c^2, one could expect that the best agreement with our method will be obtained for large quark masses. However, these

corrections are very different from our term (43). In particular, the correction (43) decreases with the quantum numbers n and ℓ because it decreases with the constituent mass, but the contribution (65) increases with these quantum numbers. So, our approach does not lead to a nonrelativistic limit compatible with previous works.

Actually, in order to obtain linear Regge trajectories, it is necessary to obtain $\Delta M \propto 1/M$ (see Eq. (58)) for light quark systems. But, for heavy quark systems, one could expect that $\Delta M \propto 1/m^2$ with $m \approx M/2$. Our model clearly misses this transition. A proper treatment in a covariant formalism with constraints could probably cure this flaw, but it is out of the scope of this paper.

VII. COMPARISON WITH EXPERIMENTAL DATA

We saw in Sec. VI that our formalism had unusual features, compared with already known results. So it is important to verify if it can correctly reproduce the experimental data. We will make here such an attempt with the $n\bar{n}$ and $b\bar{b}$ mesons, in order to check our model in different mass domains. The ingredients we put in our "realistic" model are: The Hamiltonian (37) including the string correction [15, 16], the exact retardation term ΔM, the effective Coulomb potential (36) treated as a perturbation and the quark self-energy [20].

The string correction has the following form

$$\Delta M_{\text{string}} = -\frac{a\ell(\ell+1)\langle 1/r\rangle}{\langle\mu\rangle(6\langle\mu\rangle + a\langle r\rangle)}. \tag{66}$$

The quark self-energy is due to the color magnetic moment of the quark propagating through the vacuum background field, and it has been shown that it brings a contribution to the meson mass given in the symmetrical case by

$$\Delta M_{\text{QSE}} = -\frac{f\,a}{\pi}\frac{\eta(m/\delta)}{\langle\mu\rangle}, \tag{67}$$

with $f \in [3,4]$ and $\delta \in [1.0, 1.3]$ GeV. The function η is given, for instance, in Ref. [16], in which a more detailed discussion about the quark self-energy and its consequences on the meson spectrum can be found. Let us note that $\eta(0) = 1$.

The physical parameters we use are given in Table III. We have tried as much as possible to choose standard values: $a = 0.192$ GeV^2 and $\alpha_S = 0.4$ are widely used, and $m_b =$

TABLE III: Our set of physical parameters.

$a = 0.192$ GeV^2	$\alpha_S = 0.4$
$m_n = 0$	$\delta = 1.0$ GeV
$m_b = 4.660$ GeV	$f = 3.0$
	$\lambda = 1.0$ GeV

4.660 GeV is an acceptable value for the b quark. The parameter f is fixed at 3, which is the value obtained by simulations in unquenched lattice QCD calculations [21]. The value $\delta = 1.0$ GeV is used, but this choice has a very little influence [16]. Finally, we fix $\lambda = 1.0$ GeV in order to find, with the above parameters, the $n\bar{n}$ ground state near the center of gravity of the π and ρ mesons, at 612.5 MeV (see below).

Since our model includes neither the spin (S) nor the isospin (I) of the mesons, the experimental data chosen are the spin and isospin averaged masses for the light mesons, denoted M_{av} (see Table IV). These are given by [22]

$$M_{\text{av}} = \frac{\sum_{I,J}(2I+1)(2J+1)M_{I,J}}{\sum_{I,J}(2I+1)(2J+1)}, \tag{68}$$

with $\vec{J} = \vec{L} + \vec{S}$. $M_{I,J}$ are the different masses of the states with the same orbital angular momentum ℓ. For the $b\bar{b}$ mesons, we present results for the radial excitation of the Υ (see Table V). These data are taken from Ref. [23].

TABLE IV: Comparison between the spin averaged masses M_{av} of some $n\bar{n}$ family states (see Ref. [16] for more details) and the numerically computed masses M (24) of our model. The first three columns present the different states used to compute the spin averaged masses. The last column gives the contribution of the effective Coulomb term.

Family	$N^{2S+1}L_J$	M_{av} (GeV)	M (GeV)	$\langle \tilde{V}_C \rangle$ (MeV)
ρ	$1^{2S+1}S_J$	0.613 ± 0.011	0.631	-19
$a_2(1320)$	$1^{2S+1}P_J$	1.265 ± 0.011	1.235	-6
$\rho(1700)$	$1^{2S+1}D_J$	1.676 ± 0.012	1.669	-3
$a_4(2040)$	$1^{2S+1}F_J$	2.015 ± 0.012	2.022	-1

We see in Table IV that our results are in good agreement with the spin averaged masses.

For all states, the relative error is below 3%. In each case, the influence of the effective Coulomb term is very small and could be neglected. Its role in lowering the mass is played by the contribution of the retardation. Despite this unusual effect, our approach allows to correctly reproduce the spin averaged masses. With a smaller value for the parameter λ, the contribution of the Coulomb term could be enhanced, but probably to values below those obtained in other potential models.

As it can be seen in Table V, the relative error between the data and our result is below 1% for the mesons Υ. As expected in this case since β is larger, the contribution of the Coulomb potential is larger. Heavy quark systems being more sensitive to the very short range part of the interaction, better results could probably be obtained by using a running coupling constant $\alpha_S(r)$. But this is out of the scope of this paper.

TABLE V: Same as in Table IV, but for the masses $M_{\rm exp}$ of some $b\bar{b}$ mesons. The experimental error bars are smaller than 10 MeV and are not indicated.

State	$N^{2S+1}L_J$	$M_{\rm exp}$ (GeV)	M (GeV)	$\langle \tilde{V}_C \rangle$ (MeV)
$\Upsilon(1S)$	1^3S_1	9.460	9.582	-87
$\Upsilon(2S)$	2^3S_1	10.023	9.990	-52
$\Upsilon(3S)$	3^3S_1	10.355	10.294	-40
$\Upsilon(4S)$	4^3S_1	10.580	10.555	-33
$\Upsilon(10865)$	5^3S_1	10.865	10.788	-29
$\Upsilon(11020)$	6^3S_1	11.019	11.002	-26

VIII. CONCLUSION

In this paper, the retardation effects in mesons are taken into account by the introduction of a non zero relative time in the rotating string Hamiltonian [7, 8], following a procedure inspired by the covariant oscillator quark model [3, 4]. Treated as a perturbation, the part of the total Hamiltonian containing the retardation terms is a harmonic oscillator in the relative time variable, with an effective reduced mass and an effective restoring force both depending on eigenstates of the Hamiltonian independent of the relative time. The fundamental state of this oscillator gives the contribution of the retardation to the masses

as well as the relative time part of the wave function. The introduction of the retardation also affects the Coulomb part of the interaction, which is replaced by an effective damped potential. Systems containing two particles with the same mass are only considered, but our approach leads to several interesting results.

In the light quark sector, the contribution of the retardation is not negligible (around 200 MeV for massless quarks) but it is small enough to justify a perturbative treatment. Within this approach, the Coulomb interaction is strongly reduced but the meson masses are lowered by the contribution of the retardation. The linearity of the Regge trajectories is preserved, which is the most important feature of our model. At last, the relative time wave function is a gaussian function centered around zero, which confirms the validity of the equal time ansatz in first approximation.

When the quark mass increases, the contribution of the retardation to the meson masses decreases slowly. The relative time wave function becomes more and more peaked around zero, as expected. Unfortunately, our model does not lead to a nonrelativistic limit in agreement with previous works. This is probably due to the treatment of the relative time which is not compatible with a proper elimination of this unphysical degree of freedom [1, 2].

Nevertheless, when the effective Coulomb potential and the quark self-energy [16] are included in our rotating string model with the retardation effects, meson spectra can be computed in good agreement with the experimental data.

This work must be considered as a trial to compute easily the retardation effects in mesons. The simplified and approximate approach used here has no firm theoretical basis, but it shows that the contribution of these mechanisms to the masses could be non negligible. Moreover, the importance of the retardation correction are not controlled by the quark mass m but by its constituent–state dependent–mass $\langle\sqrt{\vec{p}^{\,2}+m^2}\rangle$ [19].

APPENDIX: COEFFICIENTS OF THE LAGRANGIAN

The coefficients used in the Lagrangian (2) are defined by

$$a_1 = \mu_1 + \mu_2 + \int_0^1 d\beta\, \nu, \tag{A.1a}$$

$$a_2 = \mu_1 - \zeta(\mu_1 + \mu_2) + \int_0^1 d\beta\, (\beta - \zeta)\, \nu, \tag{A.1b}$$

$$a_3 = \mu_1(1-\zeta)^2 + \mu_2\zeta^2 + \int_0^1 d\beta\, (\beta - \zeta)^2\, \nu, \tag{A.1c}$$

$$a_4 = \int_0^1 d\beta \left(\eta^2\nu - \frac{a^2}{\nu}\right), \tag{A.1d}$$

$$c_1 = \int_0^1 d\beta\, \eta\, \nu, \tag{A.1e}$$

$$c_2 = \int_0^1 d\beta\, (\beta - \zeta)\, \eta\, \nu. \tag{A.1f}$$

Using the extremal values (14a) and (14b) of the auxiliary fields η and ν, we can compute the following relations

$$\int_0^1 d\beta\nu = \frac{ar}{y_t}\left[\arcsin s\right]_{-y_2}^{y_1}, \tag{A.2a}$$

$$\int_0^1 \frac{d\beta}{\nu} = \frac{1}{2ary_t}\left[s\sqrt{1-s^2} + \arcsin s\right]_{-y_2}^{y_1}, \tag{A.2b}$$

$$\int_0^1 d\beta(\beta-\zeta)\nu = -\frac{ar}{y_t^2}\left[\sqrt{1-s^2}\right]_{-y_2}^{y_1}, \tag{A.2c}$$

$$\int_0^1 d\beta(\beta-\zeta)^2\nu = \frac{ar}{2y_t^3}\left[-s\sqrt{1-s^2} + \arcsin s\right]_{-y_2}^{y_1}. \tag{A.2d}$$

In the symmetrical case, we have $\mu_1 = \mu_2 = \mu$, $m_1 = m_2 = m$, $y_1 = y_2 = y$, $y_t = 2y$, $\zeta = 1/2$, $\phi = 1/2$, and $\tilde{\mu} = \mu/2$. Equations (A.1) are in this case given by

$$a_1 = 2\mu + \frac{ar}{y}\arcsin y, \tag{A.3a}$$

$$a_2 = 0, \tag{A.3b}$$

$$a_3 = \frac{\mu}{2} + \frac{ar}{8y^3}\left(-y\sqrt{1-y^2} + \arcsin y\right), \tag{A.3c}$$

$$a_4 = -\frac{a}{2ry}\left(y\sqrt{1-y^2} + \arcsin y\right) + \kappa^2\frac{ar}{8y^3}\left(-y\sqrt{1-y^2} + \arcsin y\right), \tag{A.3d}$$

$$c_1 = -\kappa\frac{ar}{2y^2}\sqrt{1-y^2}, \tag{A.3e}$$

$$c_2 = \kappa\frac{ar}{8y^3}\left(-y\sqrt{1-y^2} + \arcsin y\right). \tag{A.3f}$$

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